IA Groups – Example Sheet 4

Michaelmas 2023 hjrw2@cam.ac.uk

Questions marked * are more challenging. As usual, 'identify' means 'find a standard group that it is isomorphic to'.

1. Let K be a normal subgroup of order 2 in a group G. Show that K is a subgroup of the centre $Z(G)$ of G.

2. Show that any proper subgroup of A_5 has index greater than 4.

3. Let $G \subseteq SL_3(\mathbb{R})$ be the subset of all matrices of the form
$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$
Prove that G is a subgroup. Let H be the subset of those matrices with $a = c = 0$. Show that H is a normal subgroup of G, and identify the quotient group G/H.

4. Let $G \subseteq SL_3(\mathbb{R})$ be the subset of all matrices of the form
$$\begin{pmatrix} a & 0 & 0 \\ b & c & d \\ e & f & g \end{pmatrix}.$$
Prove that G is a subgroup. Construct a surjective homomorphism $\phi : G \to GL_2(\mathbb{R})$, and identify its kernel.

5. Show that matrices $A, B \in SL_2(\mathbb{C})$ are conjugate in $SL_2(\mathbb{C})$ if and only if they are conjugate in $GL_2(\mathbb{C})$. Show that conjugate matrices in $SL_2(\mathbb{C})$ have the same trace. Conversely, show that if $\operatorname{tr}(A) = \operatorname{tr}(B)$, then A and B are conjugate in $SL_2(\mathbb{C})$ unless $\operatorname{tr}(A) = \pm 2$. Give examples to show that the result does not extend to the cases when $\operatorname{tr}(A) = \pm 2$.

6. Let $SL_2(\mathbb{R})$ act on $\mathbb{C}_\infty$ by Möbius transformations. Find the orbits and identify the stabilisers of both i and ∞. By considering the orbit of i under the action of the stabiliser of ∞, show that every $g \in SL_2(\mathbb{R})$ can be written as $g = hk$ with h upper triangular and $k \in SO(2)$. In how many ways can this be done?

7. Suppose that N is a normal subgroup of $O(2)$. Show that if N contains a reflection then $N = O(2)$.

8. Which pairs of elements of $SO(3)$ commute?

9. If $A \in M_n(\mathbb{C})$ with entries A_{ij}, let $A^\dagger \in M_n(\mathbb{C})$ have entries $\overline{A_{ji}}$. A matrix is called *unitary* if $AA^\dagger = I_n$. Show that the set $U(n)$ of unitary matrices is a subgroup of $GL_n(\mathbb{C})$. Show that
$$SU(n) = \{A \in U(n) \mid \det A = 1\}$$
is a normal subgroup of $U(n)$ and that $U(n)/SU(n) \cong S^1$. Show that Q_8 is isomorphic to a subgroup of $SU(2)$.

10. Show that if n is odd then $O(n) \cong SO(n) \times C_2$. Is $SO(2)$ a factor of a direct-product decomposition of $O(2)$? * Is there any even n such that $SO(n)$ is a factor of a direct-product decomposition of $O(n)$?

11. Let $X = \{B \in M_2(\mathbb{R}) \mid \operatorname{tr}(B) = 0\}$. Show that $SL_2(\mathbb{R})$ acts by conjugation on X. Find the orbit and stabiliser of
$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$
Show that the set Y of matrices in X with determinant 0 is the union of three orbits.

12. * Does $GL_2(\mathbb{R})$ have a subgroup isomorphic to Q_8?

13. * Let G be a finite non-trivial subgroup of $SO(3)$. Let

$$X = \{v \in \mathbb{R}^3 \mid |v| = 1 \text{ and } \mathrm{Stab}_G(v) \neq 1\}.$$

Show that G acts on X and that there are either 2 or 3 orbits. Identify G if there are 2 orbits. Find examples of such subgroups G with three orbits.